StudentID: ______________________

PHYS 451 Quantum Mechanics II (Fall 2017)
Instructor: Sergiy Bubin
Final Exam

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. Consider a hydrogen atom with a simplified form for the hyperfine interaction

$$H = H_0 + \lambda(\mathbf{S}_p \cdot \mathbf{S}_e),$$

where $\mathbf{S}_p$ and $\mathbf{S}_e$ are the operators representing the spin of the proton and electron respectively, and H_0 is the spin-independent part of the Hamiltonian. Now, assume that initially, at $t = 0$, the atom is in the ground state of H_0 and with the proton spin up and the electron spin down, i.e. in the state $\psi_{100}\chi_+^{(p)}\chi_-^{(e)}$. Do not assume that λ is small.

(a) What is the wave function of the system at $t > 0$?

(b) What is the probability of finding the spin of the proton pointing down?

Problem 2. Consider a 1D harmonic oscillator of mass m and frequency ω. Find the leading relativistic correction to its ground state energy.

Problem 3. Consider a four-level system with the Hamiltonian

$$H = \epsilon \begin{pmatrix} 1 & 1 & 0 & \gamma \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \gamma & 0 & 1 & 1 \end{pmatrix},$$

where ϵ and γ are some constants, and $\gamma \ll 1$. Using the perturbation theory find the energy levels of the system up to the second order in γ.

Problem 4. Using the Rayleigh-Ritz variational method estimate the energies of the ground and first excited states of a particle of mass m moving in the potential $V(x) = \beta x^4$. As a variational basis employ the lowest two states of 1D harmonic oscillator with varying width of the Gaussians.

Problem 5. A particle is incident on a central potential $V(r)$, which is infinitely high at $r \leq a$ and vanish at $r > a$, i.e.

$$V(r) = \begin{cases} 0, & r > a \\ \infty, & r \leq a \end{cases}$$

Find the total cross section when the energy of the incident particle is low. Define what "low" means in this context.

Problem 6. Consider a system with Hamiltonian H, which has no explicit dependence on time. Initially, at $t = 0$, the system is in the state $|\psi(0)\rangle = |\psi_i\rangle$. Show that for small time interval t the probability of finding the system in the initial state is equal to

$$1 - \Delta E^2 t^2/\hbar^2 + \mathcal{O}(t^4),$$

where ΔE is the energy uncertainty in the initial state.

Appendix: formula sheet

Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$ $(0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\,(2\alpha x^2 - 1)\,e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

$$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\,\delta_{nk}\right)$$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} - \sqrt{n}\,\delta_{k,n-1}\right)$

Creation and annihilation operators for harmonic oscillator

$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$ $\hat{N} = \hat{a}^\dagger\hat{a}$ $[\hat{a},\hat{a}^\dagger] = 1$

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R_{nl}}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom

$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2},$

The few first radial wave functions R_{nl} for the hydrogen atom ($a = \frac{4\pi\epsilon_0\hbar^2}{mZe^2}$)

$R_{10} = 2a^{-3/2}\, e^{-\frac{r}{a}} \qquad R_{20} = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2}\frac{r}{a}\right) e^{-\frac{r}{2a}} \qquad R_{21} = \frac{1}{\sqrt{24}} a^{-3/2}\, \frac{r}{a}\, e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}}\, \frac{z}{r} \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}}\, \frac{x \pm iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$\hat{\mathbf{L}}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

Fundamental commutation relations for the components of angular momentum

$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z \qquad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$

Raising and lowering operators for the z-projection of the angular momentum

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ Action: $\hat{J}_\pm|j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\, |j, m \pm 1\rangle$

Pauli matrices

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Matrix form of angular momentum operators for $l = 1$

$L_x = \frac{1}{\sqrt{2}}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_y = \frac{1}{\sqrt{2}}\hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta j_1 and j_2

$$|J\, M\, j_1\, j_2\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1\, m_1\, j_2\, m_2 | J\, M\, j_1\, j_2\rangle\, |j_1\, m_1\rangle\, |j_2\, m_2\rangle \qquad m_1 + m_2 = M$$

$$|j_1\, m_1\rangle\, |j_2\, m_2\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2} \langle J M j_1 j_2 | j_1 m_1 j_2 m_2\rangle\, |J\, M\, j_1\, j_2\rangle \qquad M = m_1 + m_2$$

Electron in a magnetic field

Hamiltonian: $H = -\boldsymbol{\mu} \cdot \mathbf{B} = -\gamma\, \mathbf{B} \cdot \mathbf{S} = \frac{e}{m}\, \mathbf{B} \cdot \mathbf{S} = \mu_{\mathrm{B}}\, \mathbf{B} \cdot \boldsymbol{\sigma}$
here $e > 0$ is the magnitude of the electron electric charge and $\mu_{\mathrm{B}} = \frac{e\hbar}{2m}$

Bloch theorem for periodic potentials $V(x + a) = V(x)$

$\psi(x) = e^{ikx}u(x)$, where $u(x + a) = u(x)$ Equivalent form: $\psi(x + a) = e^{ika}\psi(x)$

Density matrix $\hat{\rho}$

$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $\sum_i p_i = 1$
Expectation value of some observable A: $\langle\hat{A}\rangle = \sum_i p_i \langle\psi_i|\hat{A}|\psi_i\rangle = \mathrm{tr}(\hat{\rho}\hat{A})$, where $\mathrm{tr}(\hat{\rho}) = 1$

Time evolution operator

$$\hat{U}(t_f,t_i)=\hat{\mathcal{T}}\exp\left[-\frac{i}{\hbar}\int_{t_i}^{t_f}\hat{H}(t)dt\right]=1+\sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_i}^{t_f}dt_1\int_{t_i}^{t_1}dt_2\ldots\int_{t_i}^{t_{n-1}}dt_n\hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n)$$

In particular, $\hat{U}(t_f,t_i)=\exp\left[-\frac{i}{\hbar}\hat{H}(t_f-t_i)\right]$ when $\hat{H}\neq\hat{H}(t)$

Schrödinger, Heisenberg and interaction pictures

$$\psi_H=\hat{U}^{-1}\psi_S,\ \ \psi_H=\psi_S(t{=}0),\ \ \hat{A}_H=\hat{U}^{-1}\hat{A}_S\hat{U},\ \ i\hbar\frac{\hat{A}_H}{dt}=[\hat{A}_H,\hat{H}]+i\hbar\frac{\partial\hat{A}_H}{\partial t},\ \ \frac{\partial\hat{A}_H}{\partial t}\equiv\hat{U}^{-1}\frac{\partial\hat{A}_S}{\partial t}\hat{U}$$

If $\hat{H}=\hat{H}_0+\hat{V}(t)$, then

$$\psi_I=\hat{U}_0^{-1}\psi_S,\ \ \hat{U}_0=\exp\left[-\frac{i}{\hbar}\hat{H}_0t\right],\ \ \hat{A}_I=\hat{U}_0^{-1}\hat{A}_S\hat{U}_0,\ \ i\hbar\frac{\partial\hat{\psi}_I}{\partial t}=\hat{V}_I\psi_I$$

$$\psi_I(t)=\psi_I(0)+\frac{1}{i\hbar}\int\limits_0^t\hat{V}_I(t')\psi_I(t')dt'$$

Rayleigh-Ritz variational method

$$\psi_{\text{trial}}=\sum_{i=1}^n c_i\phi_i\qquad Hc=\epsilon Sc,\qquad \text{where } c=\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}\qquad\text{and}\qquad \begin{matrix}H_{ij}=\langle\phi_i|\hat{H}|\phi_j\rangle\\ S_{ij}=\langle\phi_i|\phi_j\rangle\end{matrix}$$

Stationary perturbation theory formulae

$$H=H^0+\lambda H',\qquad E_n=E_n^{(0)}+\lambda E_n^{(1)}+\lambda^2E_n^{(2)}+\ldots,\qquad \psi_n=\psi_n^{(0)}+\lambda\psi_n^{(1)}+\lambda^2\psi_n^{(2)}+\ldots$$

$$E_n^{(1)}=H'_{nn}$$

$$\psi_n^{(1)}=\sum_m c_{nm}\psi_m^{(0)},\qquad c_{nm}=\begin{cases}\frac{H'_{mn}}{E_n^{(0)}-E_m^{(0)}}, & n\neq m\\ 0, & n=m\end{cases}$$

$$E_n^{(2)}=\sum_{m\neq n}\frac{|H'_{mn}|^2}{E_n^{(0)}-E_m^{(0)}}$$

$$\psi_n^{(2)}=\sum_m d_{nm}\psi_m^{(0)},\qquad d_{nm}=\begin{cases}\frac{1}{E_n^{(0)}-E_m^{(0)}}\left(\sum\limits_{k\neq n}\frac{H'_{mk}H'_{kn}}{E_n^{(0)}-E_k^{(0)}}\right)-\frac{H'_{nn}H'_{mn}}{\left(E_n^{(0)}-E_m^{(0)}\right)^2}, & n\neq m\\ 0, & n=m\end{cases}$$

Bohr-Sommerfeld quantization rules

$\int\limits_a^b p(x)dx=(n-\frac{1}{2})\pi\hbar$ – the potential has no vertical walls at a or b

$\int\limits_a^b p(x)dx=(n-\frac{1}{4})\pi\hbar$ – only one wall of the potential is vertical

$\int\limits_a^b p(x)dx=n\pi\hbar$ – both walls of the potential are vertical

Here a and b are classical turning points and $n=1,2,3,\ldots$

Semiclassical barrier tunneling

$$T\sim\exp\left[-2\int\limits_a^b\kappa(x)dx\right]\qquad \kappa(x)=\frac{1}{\hbar}\sqrt{2m(V(x)-E)}$$

General time-dependence of the wave function (TDSE in matrix form)

$$H(\mathbf{r},t) = H^0(\mathbf{r}) + \lambda H'(\mathbf{r},t), \qquad H^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}},$$

$$i\hbar\frac{dc_n(t)}{dt} = \lambda \sum_k H'_{nk}e^{i\omega_{nk}t}c_k(t), \qquad H'_{nk} = \langle\phi_n|H'|\phi_k\rangle, \qquad \omega_{nk} = \frac{E_n^{(0)} - E_k^{(0)}}{\hbar}$$

Time-dependent perturbation theory formulae

$H(\mathbf{r},t) = H^0(\mathbf{r}) + \lambda H'(\mathbf{r},t), \qquad H^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \lambda H'$ is small

$$\psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}}, \qquad c_n(t) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \dots$$

If $c_n(t_0) = \delta_{nm}$ then at time $t > t_0$

$$c_n^{(0)} = \delta_{nm},$$

$$c_n^{(1)}(t) = \frac{1}{i\hbar}\int_{t_0}^{t} H'_{nm}(t')\, e^{i\omega_{nm}t'}dt',$$

$$c_n^{(2)}(t) = \left(\frac{1}{i\hbar}\right)^2 \sum_k \int_{t_0}^{t} dt' \int_{t_0}^{t'} H'_{nk}(t')H'_{km}(t'')\, e^{i\omega_{nk}t'}e^{i\omega_{km}t''}dt'', \qquad \dots$$

Fermi's golden rule

Transition probability: $P_{i\to f}(t) = \frac{2\pi t}{\hbar}|\mathcal{H}'_{fi}|^2\, g(E_f)$, Transition rate: $\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\mathcal{H}'_{fi}|^2\, g(E_f)$
where $\mathcal{H}'_{fi} = \langle\varphi_f|\mathcal{H}'(\mathbf{r})|\varphi_i\rangle$ and $g(E)$ is the density of states

Stationary quantum scattering

Wave function at $r \to \infty:\ \psi(r,\theta,\phi) \approx A\left[e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}\right], \quad k = \frac{\sqrt{2mE}}{\hbar}$

Differential cross section: $\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$ Total cross section: $\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega}d\Omega$

Partial wave analysis

For a spherically symmetric potential $\psi(r,\theta) = A\left[e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)a_l\, h_l^{(1)}(kr)P_l(\cos\theta)\right]$

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l\, P_l(\cos\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$$

$$\sigma_{\text{tot}} = 4\pi\sum_{l=0}^{\infty}(2l+1)|a_l|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

Relation between partial wave amplitudes and phase shifts: $a_l = \frac{1}{k}e^{i\delta_l}\sin\delta_l$

Rayleigh formula for a plane wave expansion: $e^{ikz} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta)$

Lippmann-Schwinger equation

$$\psi(\mathbf{r}) = \varphi(\mathbf{r}) + \frac{2m}{\hbar^2}\int G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}',$$

where $\varphi(\mathbf{r})$ – free-particle solution (incident wave), $G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$ – Green's function

Born approximation

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')d\mathbf{r}', \qquad \mathbf{q} = \mathbf{k}' - \mathbf{k}, \qquad q = 2k\sin\frac{\theta}{2}, \qquad \mathbf{k} = k\hat{\mathbf{r}}, \qquad \mathbf{k}' = k\hat{\mathbf{z}}$$

For spherically symmetric potentials $f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^{\infty} rV(r)\sin(qr)dr$

Adiabatic evolution of a particle that starts in the k-th state of a time-dependent Hamiltonian $\hat{H}(t)$

$$\Psi_k(\mathbf{r},t) = e^{i\theta_k(t)}e^{i\gamma_k(t)}\psi_k(\mathbf{r},t), \qquad \hat{H}(t)\psi_k(\mathbf{r},t) = E_k(t)\psi_k(\mathbf{r},t), \qquad \theta_k(t) = -\frac{1}{\hbar}\int_0^t E_k(t')dt',$$

$$\gamma_k(t) = i\int_0^t \langle\psi_k(\mathbf{r},t')|\frac{\partial}{\partial t'}\psi_k(\mathbf{r},t')\rangle dt' = i\int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \langle\psi_k|\nabla_{\mathbf{R}}\psi_k\rangle \cdot d\mathbf{R}, \qquad \mathbf{R}(t) = (R_1(t), R_2(t), \dots, R_N(t)),$$

$R_i(t),\ i = 1,\dots,N$ are parameters in the Hamiltinian that change with time

Dirac delta function

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk \qquad \delta(-x) = \delta(x) \qquad \delta(cx) = \frac{1}{|c|}\delta(x)$$

Fourier transform conventions

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk$$

or, in terms of $p = \hbar k$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} f(x)e^{-ipx/\hbar}dx \qquad f(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \tilde{f}(p)e^{ipx/\hbar}dp$$

Legendre polynomials

$$P_0(x) = 1,\ \ P_1(x) = x,\ \ P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},\ \ P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x,\ \ \ldots,\ \ P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

Orthogonality: $\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{ll'}$

Spherical Bessel equation

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + [k^2r^2 + l(l+1)]R(r) = 0$$

Spherical Bessel, Neumann, and Hankel functions

$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad \ldots, \quad j_l(x) = (-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\sin x}{x}$

$n_0(x) = -\frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \quad \ldots, \quad n_l(x) = -(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\cos x}{x}$

$h_l^{(1,2)}(x) = j_l(x) \pm i n_l(x)$

$h_0^{(1)}(x) = -i\frac{e^{ix}}{x}, \quad h_1^{(1)}(x) = \left(-\frac{i}{x^2} - \frac{1}{x}\right)e^{ix}, \quad h_2^{(1)}(x) = \left(-\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right)e^{ix}, \quad \ldots$

$h_0^{(2)}(x) = i\frac{e^{-ix}}{x}, \quad h_1^{(2)}(x) = \left(\frac{i}{x^2} - \frac{1}{x}\right)e^{-ix}, \quad h_2^{(2)}(x) = \left(\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right)e^{-ix}, \quad \ldots$

For $x \ll 1$: $j_l(x) \to \frac{2^l l!}{(2l+1)!}x^l, \quad n_l \to -\frac{(2l)!}{2^l l!}x^{-l-1}$

For $x \gg 1$: $h_l^{(1)} \to \frac{1}{x}(-i)^{l+1}e^{ix}, \quad h_l^{(2)} \to \frac{1}{x}(i)^{l+1}e^{-ix}$

Useful integrals

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\arctan\left[\frac{x}{\sqrt{a^2-x^2}}\right]\right)$$

$$\int_a^b \frac{1}{x}\sqrt{(x-a)(b-x)}\,dx = \frac{\pi}{2}\left(\sqrt{b}-\sqrt{a}\right)^2$$

$$\int_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int_0^\infty x^{2k+1}e^{-\beta x^2}dx = \frac{1}{2}\frac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int_0^\infty x^k e^{-\gamma x}dx = \frac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma > 0,\ k = 0,1,2,...)$$

$$\int_{-\infty}^{\infty} e^{-\beta x^2}e^{iqx}dx = \sqrt{\frac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta > 0)$$

$$\int_0^\pi \sin^{2k}x\,dx = \pi\frac{(2k-1)!!}{2^k\,k!} \quad (k = 0,1,2,...)$$

$$\int_0^\pi \sin^{2k+1}x\,dx = \frac{2^{k+1}\,k!}{(2k+1)!!} \quad (k = 0,1,2,...)$$

$$\int_0^{2\pi} \cos m\phi\, e^{in\phi}\,dx = \pi(\delta_{m,n} + \delta_{m,-n}) \quad (m,n = 0,\pm1,\pm2,...)$$

Useful Fourier integrals

$\int \frac{1}{|\mathbf{r}|} e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r} = \frac{4\pi}{|\mathbf{q}|^2}$

Useful trigonometric identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

Useful identities for hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1$ $\tanh^2 x + \operatorname{sech}^2 x = 1$ $\coth^2 x - \operatorname{csch}^2 x = 1$