StudentID: ____________________

PHYS 451 Quantum Mechanics II (Fall 2017)
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Midterm Exam 2

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. Consider the radial motion of an electron in the hydrogen atom (recall that the effective potential $V_{\text{eff}}(r)$ that gives rise to an infinite number of bound states includes the centrifugal term $\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$). Using the WKB approximation estimate the bound state energies of the electron. Some integral(s) in the formula sheet may be useful.

Problem 2. Consider a two-level atom with states ψ_a and ψ_b and energies $E_a < E_b$ that at $t = 0$ is in the ground state and is subjected to a harmonic perturbation $H'(x,t) = \frac{1}{2}ex\mathcal{E}(e^{i\omega t} + e^{-i\omega t})$. Without assuming smallness of H' but neglecting rapidly oscillating terms $e^{i(\omega+\omega_{ab})t}$ find the probability of transition to the excited state at $t > 0$. Make sure you apply the selection rules for the dipole matrix elements with atomic states. It may be convenient to express your result through the quantities $\Omega \equiv \frac{e\mathcal{E}}{\hbar}\langle\psi_a|x|\psi_b\rangle$, $\delta \equiv \omega - \omega_{ab}$, and $\omega_R \equiv \frac{1}{2}\sqrt{\delta^2 + \Omega^2}$.

Problem 3. A spin 1 particle is in a strong uniform magnetic field $(B_x, 0, 0)$. The gyromagnetic ratio that relates the particle's spin to its magnetic moment is g, i.e. $\boldsymbol{\mu} = g\mathbf{S}$. Initially (at $t = 0$) the particle is in the state corresponding to the zero projection of its spin on the the x-axis. A weak additional magnetic field $(0, B_y, 0)$ is suddenly turned on at time $t = 0$ and then sharply turned back off at $t = T$. What is the probability of measuring a positive or negative value of the x-projection of spin at $t > T$?

Problem 4. Consider a 1D quantum harmonic oscillator (mass m, force constant k_0) sitting in its ground state. At $t > 0$ the force constant begins changing with time as $k(t) = k_0 + \alpha t e^{-\beta t}$, where both α and β are some positive numbers and α is small. Find the probability of transition to the first, second, etc. excited states at $t = +\infty$.

Appendix: formula sheet

Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$ $(0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\,(2\alpha x^2 - 1)\,e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\,\delta_{nk}\right)$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} - \sqrt{n}\,\delta_{k,n-1}\right)$

Creation and annihilation operators for harmonic oscillator

$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$ $\hat{N} = \hat{a}^\dagger\hat{a}$ $[\hat{a},\hat{a}^\dagger] = 1$

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R_{nl}}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom

$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2},$

The few first radial wave functions R_{nl} for the hydrogen atom ($a = \frac{4\pi\epsilon_0\hbar^2}{mZe^2}$)

$R_{10} = 2a^{-3/2}\, e^{-\frac{r}{a}} \qquad R_{20} = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2}\frac{r}{a}\right) e^{-\frac{r}{2a}} \qquad R_{21} = \frac{1}{\sqrt{24}} a^{-3/2}\, \frac{r}{a}\, e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}}\, \frac{z}{r} \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}}\, \frac{x \pm iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$\hat{\mathbf{L}}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

Fundamental commutation relations for the components of angular momentum

$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z \qquad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$

Raising and lowering operators for the z-projection of the angular momentum

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ Action: $\hat{J}_\pm|j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\, |j, m \pm 1\rangle$

Pauli matrices

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Matrix form of angular momentum operators for $l = 1$

$L_x = \frac{1}{\sqrt{2}}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_y = \frac{1}{\sqrt{2}}\hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta j_1 and j_2

$$|J\, M\, j_1\, j_2\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1\, m_1\, j_2\, m_2 | J\, M\, j_1\, j_2\rangle\, |j_1\, m_1\rangle\, |j_2\, m_2\rangle \qquad m_1 + m_2 = M$$

$$|j_1\, m_1\rangle\, |j_2\, m_2\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2} \langle J M j_1 j_2 | j_1 m_1 j_2 m_2\rangle\, |J\, M\, j_1\, j_2\rangle \qquad M = m_1 + m_2$$

Electron in a magnetic field

Hamiltonian: $H = -\boldsymbol{\mu} \cdot \mathbf{B} = -\gamma\, \mathbf{B} \cdot \mathbf{S} = \frac{e}{m}\, \mathbf{B} \cdot \mathbf{S} = \mu_{\mathrm{B}}\, \mathbf{B} \cdot \boldsymbol{\sigma}$
here $e > 0$ is the magnitude of the electron electric charge and $\mu_{\mathrm{B}} = \frac{e\hbar}{2m}$

Bloch theorem for periodic potentials $V(x + a) = V(x)$

$\psi(x) = e^{ikx}u(x)$, where $u(x + a) = u(x)$ Equivalent form: $\psi(x + a) = e^{ika}\psi(x)$

Density matrix $\hat{\rho}$

$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $\sum_i p_i = 1$
Expectation value of some observable A: $\langle\hat{A}\rangle = \sum_i p_i \langle\psi_i|\hat{A}|\psi_i\rangle = \mathrm{tr}(\hat{\rho}\hat{A})$, where $\mathrm{tr}(\hat{\rho}) = 1$

Time evolution operator

$$\hat{U}(t_f,t_i)=\hat{\mathcal{T}}\exp\left[-\frac{i}{\hbar}\int_{t_i}^{t_f}\hat{H}(t)dt\right]=1+\sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_i}^{t_f}dt_1\int_{t_i}^{t_1}dt_2\ldots\int_{t_i}^{t_{n-1}}dt_n\hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n)$$

In particular, $\hat{U}(t_f,t_i)=\exp\left[-\frac{i}{\hbar}\hat{H}(t_f-t_i)\right]$ when $\hat{H}\neq\hat{H}(t)$

Schrödinger, Heisenberg and interaction pictures

$\psi_H=\hat{U}^{-1}\psi_S,\ \psi_H=\psi_S(t{=}0),\ \hat{A}_H=\hat{U}^{-1}\hat{A}_S\hat{U},\ i\hbar\frac{\hat{A}_H}{dt}=[\hat{A}_H,\hat{H}]+i\hbar\frac{\partial\hat{A}_H}{\partial t},\ \frac{\partial\hat{A}_H}{\partial t}\equiv\hat{U}^{-1}\frac{\partial\hat{A}_S}{\partial t}\hat{U}$

If $\hat{H}=\hat{H}_0+\hat{V}(t)$, then

$\psi_I=\hat{U}_0^{-1}\psi_S,\ \hat{U}_0=\exp\left[-\frac{i}{\hbar}\hat{H}_0t\right],\ \hat{A}_I=\hat{U}_0^{-1}\hat{A}_S\hat{U}_0,\ i\hbar\frac{\partial\hat{\psi}_I}{\partial t}=\hat{V}_I\psi_I$

$\psi_I(t)=\psi_I(0)+\frac{1}{i\hbar}\int\limits_0^t\hat{V}_I(t')\psi_I(t')dt'$

Rayleigh-Ritz variational method

$$\psi_{\text{trial}}=\sum_{i=1}^{n}c_i\phi_i \qquad Hc=\epsilon Sc, \qquad \text{where } c=\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix} \qquad \text{and} \qquad \begin{array}{l}H_{ij}=\langle\phi_i|\hat{H}|\phi_j\rangle\\ S_{ij}=\langle\phi_i|\phi_j\rangle\end{array}$$

Stationary perturbation theory formulae

$$H=H^0+\lambda H', \qquad E_n=E_n^{(0)}+\lambda E_n^{(1)}+\lambda^2E_n^{(2)}+\ldots, \qquad \psi_n=\psi_n^{(0)}+\lambda\psi_n^{(1)}+\lambda^2\psi_n^{(2)}+\ldots$$

$$E_n^{(1)}=H'_{nn}$$

$$\psi_n^{(1)}=\sum_m c_{nm}\psi_m^{(0)}, \quad c_{nm}=\begin{cases}\frac{H'_{mn}}{E_n^{(0)}-E_m^{(0)}}, & n\neq m\\ 0, & n=m\end{cases}$$

$$E_n^{(2)}=\sum_{m\neq n}\frac{|H'_{mn}|^2}{E_n^{(0)}-E_m^{(0)}}$$

$$\psi_n^{(2)}=\sum_m d_{nm}\psi_m^{(0)}, \quad d_{nm}=\begin{cases}\frac{1}{E_n^{(0)}-E_m^{(0)}}\left(\sum\limits_{k\neq n}\frac{H'_{mk}H'_{kn}}{E_n^{(0)}-E_k^{(0)}}\right)-\frac{H'_{nn}H'_{mn}}{\left(E_n^{(0)}-E_m^{(0)}\right)^2}, & n\neq m\\ 0, & n=m\end{cases}$$

Bohr-Sommerfeld quantization rules

$\int\limits_a^b p(x)dx=(n-\frac{1}{2})\pi\hbar$ – the potential has no vertical walls at a or b

$\int\limits_a^b p(x)dx=(n-\frac{1}{4})\pi\hbar$ – only one wall of the potential is vertical

$\int\limits_a^b p(x)dx=n\pi\hbar$ – both walls of the potential are vertical

Here a and b are classical turning points and $n=1,2,3,\ldots$

Semiclassical barrier tunneling

$$T\sim\exp\left[-2\int\limits_a^b\kappa(x)dx\right] \qquad \kappa(x)=\frac{1}{\hbar}\sqrt{2m(V(x)-E)}$$

General time-dependence of the wave function (TDSE in matrix form)

$$H(\mathbf{r},t) = H^0(\mathbf{r}) + \lambda H'(\mathbf{r},t), \qquad H^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}},$$

$$i\hbar\frac{dc_n(t)}{dt} = \lambda\sum_k H'_{nk}e^{i\omega_{nk}t}c_k(t), \qquad H'_{nk} = \langle\phi_n|H'|\phi_k\rangle, \qquad \omega_{nk} = \frac{E_n^{(0)}-E_k^{(0)}}{\hbar}$$

Time-dependent perturbation theory formulae

$H(\mathbf{r},t) = H^0(\mathbf{r}) + \lambda H'(\mathbf{r},t), \qquad H^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \lambda H'$ is small

$\psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}}, \qquad c_n(t) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \ldots$

If $c_n(t_0) = \delta_{nm}$ then at time $t > t_0$

$$c_n^{(0)} = \delta_{nm},$$

$$c_n^{(1)}(t) = \frac{1}{i\hbar}\int_{t_0}^{t} H'_{nm}(t')\,e^{i\omega_{nm}t'}dt',$$

$$c_n^{(2)}(t) = \left(\frac{1}{i\hbar}\right)^2\sum_k\int_{t_0}^{t}dt'\int_{t_0}^{t'}H'_{nk}(t')H'_{km}(t'')\,e^{i\omega_{nk}t'}e^{i\omega_{km}t''}dt'', \qquad \ldots$$

Fermi's golden rule

Transition probability: $P_{i\to f}(t) = \frac{2\pi t}{\hbar}|\mathcal{H}'_{fi}|^2\,g(E_f)$, Transition rate: $\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\mathcal{H}'_{fi}|^2\,g(E_f)$
where $\mathcal{H}'_{fi} = \langle\varphi_f|\mathcal{H}'(\mathbf{r})|\varphi_i\rangle$ and $g(E)$ is the density of states

Dirac delta function

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk \qquad \delta(-x) = \delta(x) \qquad \delta(cx) = \frac{1}{|c|}\delta(x)$$

Fourier transform conventions

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk$$

or, in terms of $p = \hbar k$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} f(x)e^{-ipx/\hbar}dx \qquad f(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \tilde{f}(p)e^{ipx/\hbar}dp$$

Useful integrals

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\arctan\left[\frac{x}{\sqrt{a^2-x^2}}\right]\right)$$

$$\int_a^b \frac{1}{x}\sqrt{(x-a)(b-x)}\,dx = \frac{\pi}{2}\left(\sqrt{b}-\sqrt{a}\right)^2$$

$$\int_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int_0^\infty x^{2k+1}e^{-\beta x^2}dx = \frac{1}{2}\frac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int_0^\infty x^k e^{-\gamma x}dx = \frac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma > 0,\ k = 0,1,2,...)$$

$$\int_{-\infty}^{\infty} e^{-\beta x^2}e^{iqx}dx = \sqrt{\frac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta > 0)$$

$$\int_0^\pi \sin^{2k}x\,dx = \pi\frac{(2k-1)!!}{2^k\,k!} \quad (k = 0,1,2,...)$$

$$\int_0^\pi \sin^{2k+1}x\,dx = \frac{2^{k+1}\,k!}{(2k+1)!!} \quad (k = 0,1,2,...)$$

$$\int_0^{2\pi} \cos m\phi\, e^{in\phi}\,dx = \pi(\delta_{m,n} + \delta_{m,-n}) \quad (m,n = 0,\pm 1,\pm 2,...)$$

Useful trigonometric identities

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \qquad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Useful identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1 \qquad \tanh^2 x + \mathrm{sech}^2\,x = 1 \qquad \coth^2 x - \mathrm{csch}^2\,x = 1$$