PHYS 452 Quantum Mechanics II (Fall 2019)
Homework #1, due Thursday August 22 in class

Variational Method

1. The deuteron is a bound state of a proton and a neutron that are attracted to each other by nuclear forces. The nuclear forces are complicated and have a short range, i.e. they fall rapidly (exponentially) at large distances. Moreover, they are known to be spin-dependent and contain a tensor component, i.e. they are not central. For certain applications, however, we could use a reasonably good central approximation for the internucleon potential: $V(r) = -V_0 e^{-r/a}$, where a is of the order of 2.2 fm and V_0 is of the order of 32 MeV (fermis and MeVs are typical units of length and energy used in nuclear physics). Using the variational method with the trial wave function in the form $\psi(r) = Ae^{-\gamma r/a}$, where γ is an adjustable parameter, determine if the above model potential supports a bound state. If it does then find the ground state energy and compare it to the value of V_0. Based on the latter comparison conclude if the deuteron is a strongly or weakly bound system. At some point in your calculation you may need to solve a simple algebraic equation numerically. Use your favorite software for that purpose or simply do it graphically (but do make sure that your answer is accurate to at least three figures).

2. Consider a 1D system with a potential that depends on the fourth power of x:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + bx^4. \tag{1}$$

First, introduce the dimensionless distance $y = \frac{x}{a}$ and get rid of $\hbar$, m, and b in the Hamiltonian (in other words, introduce natural units in which $\hbar = m = b = 1$). Then approximate the ground state wave function with the following gaussians expansion:

$$\psi(y) = \sum_{i=1}^{N} c_i e^{-\alpha_i y^2}, \tag{2}$$

where $N = 3$, $\alpha_1 = 0.5$, $\alpha_2 = 1$, and $\alpha_3 = 1.5$, while c_i's are unknown expansion coefficients (linear variational parameters). Find the corresponding energy. Is it possible to meaningfully approximate the wave function of an excited state with this expansion? If so, which excited state (i.e. corresponding to which quantum number)? To answer that question it is helpful to think about the symmetry of the trial wave function (2) as well as the symmetry of the exact eigenstates of the Hamiltonian (1). Compute the corresponding approximate energy of the excited state.

In this problem you will need to find the solutions of a generalized eigenvalue problem with $N \times N$ matrices. To do so, write out those matrices explicitly and solve the generalized eigenvalue problem numerically using any package that can do nimerical calculations (e.g. MATHEMATICA, MAPLE, MATLAB, OCTAVE, PYTHON, etc.). For example, in MATHEMATICA there is a command `Eigensystem[{H,S}]` that finds generalized eigenvalues and eigenvectors. In PYTHON you can use function `scipy.linalg.eigh` from library SciPy.