StudentID: ________________________

PHYS 452 Quantum Mechanics II (Fall 2019)
Instructor: Sergiy Bubin
Midterm Exam 2

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. Consider an extremely slow incident particle of mass m that travels a very long distance and then encounters a potential barrier in the form $V(x) = V_0 e^{-\alpha|x|}$, where V_0 and α are some positive constants. Estimate the probability that the particle tunnels through the barrier. What are the contstraints on the values of V_0 and α so that your estimate remains meaningful?

Problem 2. Apply the WKB approximation to a particle of mass m moving in the following "half-oscillator" potential

$$V(x) = \begin{cases} \frac{m\omega^2 x^2}{2}, & x \geq 0 \\ \infty, & x < 0 \end{cases}.$$

Find the energy eigenvalues and wave functions. Compare the WKB energy eigenvalues with the exact ones for this potential. Are they close to each other? Why yes or why not?

Problem 3. Consider a positively charged spin 1/2 particle in an external magnetic field $\mathbf{B}$ that is governed by the following Hamiltonian:

$$H = \varepsilon I - \gamma \mathbf{B}\cdot\mathbf{S},$$

where $\mathbf{S}$ is the spin operator, γ is the particle's gyromagnetic ratio, ε is some constant, and I is the identity operator in spin space.

(a) If the magnetic field is constant in time and given by $\mathbf{B} = B\mathbf{e}_z$ ($\mathbf{e}_z$ is a unit vector along the z-axis), determine the possible energies and corresponding eigenstates of the system

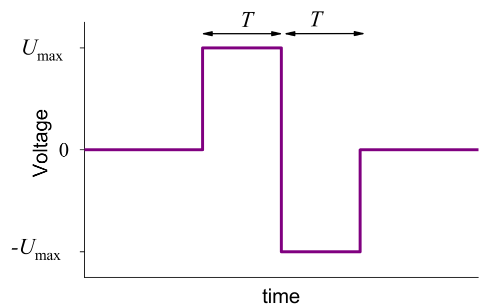
(b) Now assume that the magnetic field is time-dependent and is given by

$$\mathbf{B} = \begin{cases} B\mathbf{e}_z, & t < 0 \\ \beta(\mathbf{e}_x \cos\omega t - \mathbf{e}_y \sin\omega t) + B\mathbf{e}_z, & t \geq 0 \end{cases},$$

where β and ω are real constants. At $t = 0$ the particle is in the spin-up state. Using the time-dependent perturbation theory find the probability that the particle undergoes a transition to the spin-down state at some later time $t > 0$.

(c) For what range of values of ω your result in (b) remains valid?

Problem 4. Consider a 1D quantum harmonic oscillator of mass m, charge q, and frequency ω prepared in its ground state ($n = 0$) and placed inside a parallel plate capacitor. The separation between the plates of the capacitor is D. The figure below shows the voltage applied to the capacitor as a function of time. Essentially we have two rectangular pulses (in opposite direction), each of duration T and amplitude $U_{\max}$ (you may assume that $U_{\max}$ is small in some sense). What is the probability that the oscillator will be found in the second ($n = 2$) excited state at $t = +\infty$?



Appendix: formula sheet

Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$ $(0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\,(2\alpha x^2 - 1)\,e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{k+1}\,\delta_{n,k+1}\right)$

$$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + (2k+1)\,\delta_{nk} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2}\right)$$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} - \sqrt{k+1}\,\delta_{n,k+1}\right)$

$$\langle\phi_n|\hat{p}^2|\phi_k\rangle = -\frac{m\hbar\omega}{2}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} - (2k+1)\,\delta_{nk} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2}\right)$$

Creation and annihilation operators for harmonic oscillator

$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$ $\hat{N} = \hat{a}^\dagger\hat{a}$ $[\hat{a},\hat{a}^\dagger] = 1$

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R_{nl}}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom

$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2}$,

The few first radial wave functions R_{nl} for the hydrogen atom ($a = \frac{4\pi\epsilon_0\hbar^2}{mZe^2}$)

$R_{10} = 2a^{-3/2}\,e^{-\frac{r}{a}}$ $\quad R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{-\frac{r}{2a}}$ $\quad R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$ $\quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$ $\quad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}}\,\frac{x\pm iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$\hat{\mathbf{L}}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$ $\quad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

Fundamental commutation relations for the components of angular momentum

$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$ $\quad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$ $\quad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$

Raising and lowering operators for the z-projection of the angular momentum

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ $\quad$ Action: $\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\,|j, m\pm1\rangle$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta j_1 and j_2

$$|J\,M\,j_1\,j_2\rangle = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\langle j_1\,m_1\,j_2\,m_2|J\,M\,j_1\,j_2\rangle\,|j_1\,m_1\rangle\,|j_2\,m_2\rangle \qquad m_1 + m_2 = M$$

$$|j_1\,m_1\rangle\,|j_2\,m_2\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2}\langle JMj_1j_2|j_1m_1j_2m_2\rangle\,|J\,M\,j_1\,j_2\rangle \qquad M = m_1 + m_2$$

Pauli matrices

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Matrix form of angular momentum operators for $l = 1$

$L_x = \frac{1}{\sqrt{2}}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad L_y = \frac{1}{\sqrt{2}}\hbar\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\quad L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Electron in a magnetic field

Hamiltonian: $H = -\boldsymbol{\mu}\cdot\mathbf{B} = -\gamma\,\mathbf{B}\cdot\mathbf{S} = \frac{e}{m}\,\mathbf{B}\cdot\mathbf{S} = \mu_\mathrm{B}\,\mathbf{B}\cdot\boldsymbol{\sigma}$
here $e > 0$ is the magnitude of the electron electric charge and $\mu_\mathrm{B} = \frac{e\hbar}{2m}$

Bloch theorem for periodic potentials $V(x+a) = V(x)$

$\psi(x) = e^{ikx}u(x)$, where $u(x+a) = u(x)$ $\quad$ Equivalent form: $\psi(x+a) = e^{ika}\psi(x)$

Density matrix $\hat{\rho}$

$\hat{\rho} = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, $\quad$ where $\sum_i p_i = 1$

Expectation value of some observable A: $\langle\hat{A}\rangle = \sum_i p_i\langle\psi_i|\hat{A}|\psi_i\rangle = \mathrm{tr}(\hat{\rho}\hat{A})$, where $\mathrm{tr}(\hat{\rho}) = 1$

Time evolution operator

$$\hat{U}(t_f,t_i)=\hat{\mathcal{T}}\exp\left[-\frac{i}{\hbar}\int_{t_i}^{t_f}\hat{H}(t)dt\right]=1+\sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_i}^{t_f}dt_1\int_{t_i}^{t_1}dt_2\ldots\int_{t_i}^{t_{n-1}}dt_n\hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n)$$

In particular, $\hat{U}(t_f,t_i)=\exp\left[-\frac{i}{\hbar}\hat{H}(t_f-t_i)\right]$ when $\hat{H}\neq\hat{H}(t)$

Schrödinger, Heisenberg and interaction pictures

$\psi_H=\hat{U}^{-1}\psi_S,\ \ \psi_H=\psi_S(t{=}0),\ \ \hat{A}_H=\hat{U}^{-1}\hat{A}_S\hat{U},\ \ i\hbar\frac{\hat{A}_H}{dt}=[\hat{A}_H,\hat{H}]+i\hbar\frac{\partial\hat{A}_H}{\partial t},\ \ \frac{\partial\hat{A}_H}{\partial t}\equiv\hat{U}^{-1}\frac{\partial\hat{A}_S}{\partial t}\hat{U}$

If $\hat{H}=\hat{H}_0+\hat{V}(t)$, then

$\psi_I=\hat{U}_0^{-1}\psi_S,\ \ \hat{U}_0=\exp\left[-\frac{i}{\hbar}\hat{H}_0t\right],\ \ \hat{A}_I=\hat{U}_0^{-1}\hat{A}_S\hat{U}_0,\ \ i\hbar\frac{\partial\hat{\psi}_I}{\partial t}=\hat{V}_I\psi_I$

$\psi_I(t)=\psi_I(0)+\frac{1}{i\hbar}\int\limits_0^t\hat{V}_I(t')\psi_I(t')dt'$

Rayleigh-Ritz variational method

$$\psi_{\text{trial}}=\sum_{i=1}^n c_i\phi_i\qquad Hc=\epsilon Sc,\qquad \text{where } c=\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}\qquad\text{and}\quad\begin{array}{l}H_{ij}=\langle\phi_i|\hat{H}|\phi_j\rangle\\S_{ij}=\langle\phi_i|\phi_j\rangle\end{array}$$

Stationary perturbation theory formulae

$$H=H^0+\lambda H',\qquad E_n=E_n^{(0)}+\lambda E_n^{(1)}+\lambda^2E_n^{(2)}+\ldots,\qquad \psi_n=\psi_n^{(0)}+\lambda\psi_n^{(1)}+\lambda^2\psi_n^{(2)}+\ldots$$

$$E_n^{(1)}=H'_{nn}$$

$$\psi_n^{(1)}=\sum_m c_{nm}\psi_m^{(0)},\quad c_{nm}=\begin{cases}\frac{H'_{mn}}{E_n^{(0)}-E_m^{(0)}}, & n\neq m\\ 0, & n=m\end{cases}$$

$$E_n^{(2)}=\sum_{m\neq n}\frac{|H'_{mn}|^2}{E_n^{(0)}-E_m^{(0)}}$$

$$\psi_n^{(2)}=\sum_m d_{nm}\psi_m^{(0)},\quad d_{nm}=\begin{cases}\frac{1}{E_n^{(0)}-E_m^{(0)}}\left(\sum\limits_{k\neq n}\frac{H'_{mk}H'_{kn}}{E_n^{(0)}-E_k^{(0)}}\right)-\frac{H'_{nn}H'_{mn}}{\left(E_n^{(0)}-E_m^{(0)}\right)^2}, & n\neq m\\ 0, & n=m\end{cases}$$

Bohr-Sommerfeld quantization rules

$\int\limits_a^b p(x)dx=(n-\frac{1}{2})\pi\hbar$ – the potential has no vertical walls at a or b

$\int\limits_a^b p(x)dx=(n-\frac{1}{4})\pi\hbar$ – only one wall of the potential is vertical

$\int\limits_a^b p(x)dx=n\pi\hbar$ – both walls of the potential are vertical

Here a and b are classical turning points and $n=1,2,3,\ldots$

Semiclassical barrier tunneling

$$T\sim\exp\left[-2\int\limits_a^b\kappa(x)dx\right]\qquad \kappa(x)=\frac{1}{\hbar}\sqrt{2m(V(x)-E)}$$

General time-dependence of the wave function (TDSE in matrix form)

$$H(\mathbf{r},t) = H^0(\mathbf{r}) + \lambda H'(\mathbf{r},t), \qquad H^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}},$$

$$i\hbar\frac{dc_n(t)}{dt} = \lambda\sum_k H'_{nk}e^{i\omega_{nk}t}c_k(t), \qquad H'_{nk} = \langle\phi_n|H'|\phi_k\rangle, \qquad \omega_{nk} = \frac{E_n^{(0)} - E_k^{(0)}}{\hbar}$$

Time-dependent perturbation theory formulae

$H(\mathbf{r},t) = H^0(\mathbf{r}) + \lambda H'(\mathbf{r},t), \qquad H^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \lambda H'$ is small

$\psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}}, \qquad c_n(t) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \dots$

If $c_n(t_0) = \delta_{nm}$ then at time $t > t_0$

$$c_n^{(0)} = \delta_{nm},$$

$$c_n^{(1)}(t) = \frac{1}{i\hbar}\int_{t_0}^{t} H'_{nm}(t')\, e^{i\omega_{nm}t'}dt',$$

$$c_n^{(2)}(t) = \left(\frac{1}{i\hbar}\right)^2 \sum_k \int_{t_0}^{t} dt' \int_{t_0}^{t'} H'_{nk}(t')H'_{km}(t'')\, e^{i\omega_{nk}t'}e^{i\omega_{km}t''}dt'', \qquad \dots$$

Fermi's golden rule

Transition probability: $P_{i\to f}(t) = \frac{2\pi t}{\hbar}|\mathcal{H}'_{fi}|^2\, g(E_f)$, Transition rate: $\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\mathcal{H}'_{fi}|^2\, g(E_f)$
where $\mathcal{H}'_{fi} = \langle\varphi_f|\mathcal{H}'(\mathbf{r})|\varphi_i\rangle$ and $g(E)$ is the density of states

Stationary quantum scattering

Wave function at $r \to \infty$: $\psi(r,\theta,\phi) \approx A\left[e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}\right], \quad k = \frac{\sqrt{2mE}}{\hbar}$

Differential cross section: $\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$ Total cross section: $\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega}d\Omega$

Partial wave analysis

For a spherically symmetric potential $\psi(r,\theta) = A\left[e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)a_l\, h_l^{(1)}(kr)P_l(\cos\theta)\right]$

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l\, P_l(\cos\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$$

$$\sigma_{\text{tot}} = 4\pi\sum_{l=0}^{\infty}(2l+1)|a_l|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

Relation between partial wave amplitudes and phase shifts: $a_l = \frac{1}{k}e^{i\delta_l}\sin\delta_l$

Rayleigh formula for a plane wave expansion: $e^{ikz} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta)$

Lippmann-Schwinger equation

$\psi(\mathbf{r}) = \varphi(\mathbf{r}) + \frac{2m}{\hbar^2}\int G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}',$

where $\varphi(\mathbf{r})$ is a free-particle solution (incident wave) and $G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$ is Green's function

Born approximation

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')d\mathbf{r}', \quad \mathbf{q} = \mathbf{k}' - \mathbf{k}, \quad q = 2k\sin\frac{\theta}{2}, \quad \mathbf{k} = k\hat{\mathbf{r}}, \quad \mathbf{k}' = k\hat{\mathbf{z}}$$

For spherically symmetric potentials $f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^{\infty} rV(r)\sin(qr)dr$

Dirac delta function

$$\int\limits_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \qquad \delta(x) = \tfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{ikx}dk \qquad \delta(-x) = \delta(x) \qquad \delta(cx) = \tfrac{1}{|c|}\delta(x)$$

Fourier transform conventions

$$\tilde{f}(k) = \tfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \tfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk$$

or, in terms of $p = \hbar k$

$$\tilde{f}(p) = \tfrac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ipx/\hbar}dx \qquad f(x) = \tfrac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{+\infty} \tilde{f}(p)e^{ipx/\hbar}dp$$

Useful integrals

$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\arctan\left[\tfrac{x}{\sqrt{a^2-x^2}}\right]\right)$$

$$\int x\sin^2 x\,dx = \tfrac{x^2}{4} - \tfrac{\cos 2x}{8} - \tfrac{x\sin 2x}{4}$$

$$\int\limits_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\tfrac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int\limits_0^\infty x^{2k+1}e^{-\beta x^2}dx = \tfrac{1}{2}\tfrac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int\limits_0^\infty x^k e^{-\gamma x}dx = \tfrac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma > 0,\ k = 0,1,2,...)$$

$$\int\limits_{-\infty}^{\infty} e^{-\beta x^2}e^{iqx}dx = \sqrt{\tfrac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta > 0)$$

$$\int\limits_0^\pi \sin^{2k} x\,dx = \pi\tfrac{(2k-1)!!}{2^k\,k!} \quad (k = 0,1,2,...)$$

$$\int\limits_0^\pi \sin^{2k+1} x\,dx = \tfrac{2^{k+1}\,k!}{(2k+1)!!} \quad (k = 0,1,2,...)$$

$$\int\limits_0^{2\pi} \cos m\phi\, e^{in\phi}\,dx = \pi(\delta_{m,n} + \delta_{m,-n}) \quad (m,n = 0,\pm 1,\pm 2,...)$$

Useful trigonometric identities

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \qquad \cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \qquad \cos\alpha\sin\beta = \tfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Useful identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1 \qquad \tanh^2 x + \mathrm{sech}^2\, x = 1 \qquad \coth^2 x - \mathrm{csch}^2\, x = 1$$