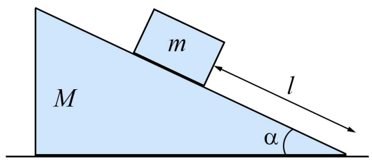
Name: ____________________

PHYS 505 - Classical Mechanics (graduate), Fall 2015
Instructor: Sergiy Bubin
Final Exam

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. should be used. Some information that you might not remember (e.g. some equation) is provided in hints.
- No communication with classmates is allowed during the test.
- Show your work, explain your reasoning. Final answers or intermediate steps without clear explanations will receive no credit.
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure problems are marked and pages are stapled together before submitting your work.

Problem 1. Consider a block of mass m that slides down the wedge of mass M without friction. The wedge can move without friction on a horizontal table. The system is initially at rest. Write the Lagrangian of the system and find the equation(s) of motion. How long will it take for the block to reach the bottom if the length of the sloping face is l?



Problem 2. Consider a pendulum in the form of a uniform rod of mass M and length l that is pivoted at one of its ends. A bug of mass $M/3$, initially at the pivot point, starts crawling down the rod with a constant speed u.

(a) Construct the Lagrangian of the rod–bug system and find the equation(s) of motion

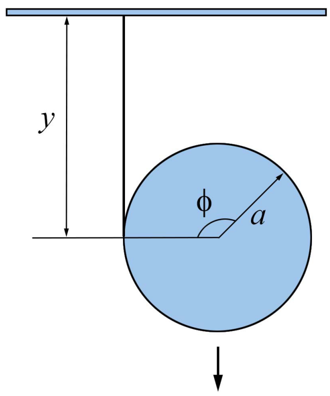
(b) Find the frequency of small oscillations when the bug has crawled distance b along the rod. Assume that the speed of the bug is small and the change in b over one oscillation period can be neglected.

Problem 3. Consider a uniform disk with a thin string wrapped around it. The end of the string is attached to a fixed support and the disk is allowed to fall with the string unwinding. Using the Lagrange undetermined multiplier method find the equation of motion of the falling disk and the force of constraint (in this system there is a constraint that connects height y and angle ϕ). What is the linear acceleration of the center of the disk, $\ddot{y}$?

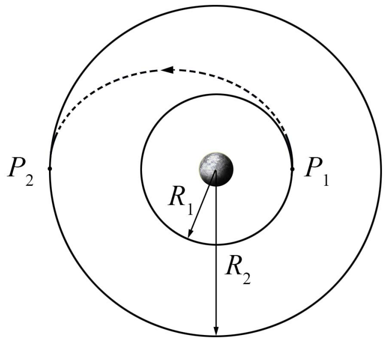
Hint: You may recall that for a system with constraints the Lagrange equations look as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k,$$

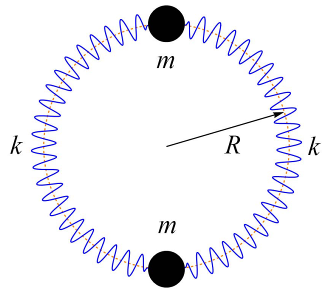
where $Q_k = \sum_i \lambda_i \frac{\partial g_i}{\partial q_k}$ *are the generalized forces of constraint,* $g_i(q_1, ..., q_n, t) = 0$ *are constraints, and* λ_i *are the Lagrange multipliers corresponding to those constraints.*



Problem 4. A spacecraft in a circular orbit of radius R_1 wishes to transfer to another circular orbit of radius R_2. It does this by two successive boosts in the velocity as shown in the figure. The first time (at point P_1) it boosts its velocity in the tangential direction so that it gets into a (temporary) elliptic orbit just large enough to take it to the required radius R_2. When reaching that required radius (at point P_2) it boosts the velocity again in such a way that it gets on a circular orbit of radius R_2. By what factor must the spacecraft change its speed in each of these two boosts (lets us call them thrust factors λ_1 and λ_2 respectively)? By what factor does the spacecraft's speed change as a result of the whole maneuver?



Problem 5. The system shown in the figure consists of two small beads of mass m. The beads are connected with two springs of force constant k and wind around a ring or radius R. The equilibrium length of each spring is πR. Find the normal frequencies and normal modes of the system. Give physical interpretation of those frequencies and modes.



Problem 6. Consider a damped harmonic oscillator,

$$\ddot{q} + 2\gamma\dot{q} + \omega^2 q = 0.$$

(a) Assuming that the Lagrangian of this system can be written as $L(\dot{q}, q, t) = f(t)\mathcal{L}(\dot{q}, q)$, where $\mathcal{L}$ is the Lagrangian of an undamped harmonic oscillator (i.e. when γ=0), find such a function $f(t)$ that reproduces the equation of motion given above. To define $f(t)$ uniquely, assume that $f(0) = 1$.

(b) Construct the Hamiltonian $H(q, p)$ using the Lagrangian and $f(t)$ found in the previous part.

(c) Do a canonical transformation using the second type of generating function, $F_2(q, P, t) = e^{\gamma t} qP$, and find the transformed Hamiltonian H'. Show that H' is conserved.

Hint: Remember that the original and transformed Hamiltonians are connected by the following relation involving the generating function F,

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - H' + \frac{dF}{dt},$$

and for the second type of canonical transformation we choose $F = F_2(q, P, t) - \sum_i Q_i P_i$