Let \( x \) be the distance of the block from the top of the wedge and \( X \) is the distance of the wedge from some point on the table.

**Kinetic energy:** 
\[
T = T_M + T_m = \frac{M}{2} \dot{X}^2 + \frac{m}{2} \left( \dot{X} + \dot{x} \cos \alpha \right)^2 + \left( \dot{x} \sin \alpha \right)^2
\]

\[
= \frac{1}{2} (M+m) \dot{X}^2 + \frac{1}{2} m \dot{x}^2 + m \dot{X} \dot{x} \cos \alpha
\]

**Potential energy:** 
\[
V = -mgx \sin \alpha
\]

**Lagrangian:** 
\[
L = \frac{1}{2} (M+m) \dot{X}^2 + \frac{1}{2} m \dot{x}^2 + m \dot{X} \dot{x} \cos \alpha + mgx \sin \alpha
\]

Lagrange equations are as follows:

\[
\ddot{X} = (M+m) \frac{\ddot{X}}{\dot{X}} + m \ddot{x} \cos \alpha = 0
\]

\[
x: \quad m \ddot{x} + m \dot{X} \cos \alpha = mg \sin \alpha
\]

From the second equation we get:
\[
\ddot{X} = \frac{1}{\cos \alpha} (g \sin \alpha - \ddot{x})
\]

which we can substitute into the first equation and obtain:
\[
(M+m)g \frac{\sin \alpha}{\cos \alpha} = \left( M + m \cos \alpha \right) \ddot{x}
\]

or:
\[
\ddot{x} = \frac{g \sin \alpha}{1 - \frac{m}{M+m} \cos^2 \alpha}
\]

Acceleration \( \ddot{x} \) is constant. Hence the time it takes to slide distance \( \ell \) is obtained from:
\[
\ell = \frac{\ddot{x} t^2}{2}
\]

so:
\[
t = \sqrt{\frac{2 \ell}{\ddot{x}}} = \sqrt{\frac{2 \ell \left( 1 - \frac{m}{M+m} \cos^2 \alpha \right)}{g \sin \alpha}}
\]
(2) a) After the bug has crawled distance \( b \), the Lagrangian of the system is

\[
L = \frac{I \dot{\theta}^2}{2} + \frac{1}{2} M u^2 + \frac{1}{2} M g l \cos \theta + \frac{1}{3} M g b \cos \theta
\]

Here \( I \) is the moment of inertia of the system that consists of the uniform rod and the bug:

\[
I = \frac{1}{3} M l^2 + \frac{1}{3} M b^2 = \frac{1}{3} M (l^2 + b^2)
\]

The Lagrange equation of motion, \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} \), becomes

\[
\frac{d}{dt}(I \dot{\theta}) = -\frac{1}{2} M g l \sin \theta - \frac{1}{3} M g b \sin \theta
\]

or

\[
I \ddot{\theta} + I \dot{\theta} = -M g \left( \frac{b}{2} + \frac{b}{3} \right) \sin \theta
\]

Now \( I = \frac{2}{3} M l b = \frac{2}{3} M b l \). Thus,

\[
\ddot{\theta} + \frac{2b}{l^2 + b^2} \dot{\theta} + \frac{3}{l^2 + b^2} g \sin \theta = 0 \hspace{1cm} \text{the equation of motion}
\]

b) Assuming \( u \) to be small we can neglect the second term in the equation of motion. We can also replace \( \sin \theta \) with \( \theta \) for small oscillations. Then we get

\[
\ddot{\theta} + \frac{3}{l^2 + b^2} g \theta = 0
\]

So the frequency of small oscillations is

\[
\omega = \sqrt{\frac{3}{l^2 + b^2} g}
\]
The kinetic energy of the disk as it falls is

\[ T = \frac{1}{2} m {\dot{y}}^2 + \frac{1}{2} I {\dot{\phi}}^2 \]

where \( m \) is the mass of the disk and \( I \) is the moment of inertia about its center of mass. \( y \) denotes the vertical position of the center of mass, while \( \phi \) is the angle of rotation about the center of mass. We know that for a uniform disk \( I = \frac{ma^2}{2} \). The potential energy (assuming the y-axis point down) is given by \( V = -mg y \). With all that our Lagrangian is

\[ L = \frac{1}{2} m {\dot{y}}^2 + \frac{1}{4} ma^2 {\dot{\phi}}^2 - mg y \]

Since the vertical position of the center of mass is related to the rotation angle as \( y = a \phi \), the constraint can be written as

\[ g(y, \phi) = y - a \phi = 0 \quad (do \ not \ confuse \ g(y, \phi) \ and \ g) \]

For a system with a constraint the Lagrange equations are

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_n} \right) - \frac{\partial L}{\partial q_n} = \lambda \frac{\partial g(y, \phi)}{\partial \dot{q}_n} \]

In our case we get

\[ y: \quad m{\ddot{y}} - mg = \lambda \]

\[ \phi: \quad \frac{1}{2} ma^2 {\ddot{\phi}} = -\lambda a \]

These two equations and the equation of constraint can be solved easily. If we differentiate the equation of constraint we get \( \ddot{\phi} = \frac{\ddot{y}}{a} \), which can be inserted into the equation for \( \phi \). That gives

\[ \begin{align*}
  m{\ddot{y}} - mg &= \lambda \\
  \frac{1}{2} ma^2 \ddot{\phi} &= -\lambda a
\end{align*} \]

The forces of constraint are

\[ Q_y = \lambda \frac{\partial g(y, \phi)}{\partial y} = \lambda = -\frac{1}{3} mg \]

\[ Q_{\phi} = \lambda \frac{\partial g(y, \phi)}{\partial \phi} = -\lambda a = \frac{1}{3} mga \]
For the circular orbits of radius $R_1$ and $R_2$ the velocity can be easily found by equating the gravitational and centrifugal forces

$$
\frac{m u_1^2}{2} = \frac{G m M}{R_1^2}
$$
$$
\frac{m u_2^2}{2} = \frac{G m M}{R_2^2}
$$

where $m$ and $M$ are the masses of the spacecraft and the Earth respectively.

Which yields $u_1 = \sqrt{\frac{GM}{R_1}}$ and $u_2 = \sqrt{\frac{GM}{R_2}}$. This immediately gives the answer to the last question — by what factor does the spacecraft velocity change in the whole maneuver:

$$
\frac{u_2}{u_1} = \sqrt{\frac{R_1}{R_2}}
$$

Now the intermediate orbit is an elliptic one. At points $P_1$ and $P_2$ (perihelion and aphelion) the radial component of the velocity is zero. Let us denote $v_p$ and $v_a$ the tangential components at the perihelion and aphelion respectively. The conservation of the angular momentum, $l = m r v_{th}$, and the total energy, $E = \frac{mv^2}{2} + V(r)$ gives

$$
\begin{cases}
  m R_1 v_p = m R_2 v_a \\
  \frac{m v_p^2}{2} - \frac{G m M}{R_1} = \frac{m v_a^2}{2} - \frac{G m M}{R_2}
\end{cases}
$$

or

$$
\begin{cases}
  R_1 v_p = R_2 v_a \\
  v_p^2 - v_a^2 = GM \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\end{cases}
$$

Solving for $v_p$ and $v_a$ yields

$$
\begin{align*}
  v_p &= \sqrt{2GM} \frac{R_2}{R_1 + R_2} \\
  v_a &= \sqrt{2GM} \frac{R_1}{R_2 + R_1 + R_2}
\end{align*}
$$

Then we can determine the required thrust factors at points $P_1$ and $P_2$:

$$
\lambda_1 = \frac{v_p}{u_1} = \sqrt{\frac{2R_2}{R_1 + R_2}}
$$
$$
\lambda_2 = \frac{v_a}{u_2} = \sqrt{\frac{2R_1}{R_1 + R_2}}
$$
If the angular positions of the beads are denoted \( \Theta_1 \) and \( \Theta_2 \) respectively, then the kinetic energy of the system is

\[
T = \frac{1}{2} m R^2 \dot{\Theta}_1^2 + \frac{1}{2} m R^2 \dot{\Theta}_2^2
\]

The potential energy, on the other hand, is

\[
V = \frac{1}{2} k R^2 (\Theta_2 - \Theta_1 - \pi)^2 + \frac{1}{2} k R^2 (\Theta_1 - \Theta_2 - \pi)^2
\]

The Lagrangian is then

\[
L = \frac{1}{2} m R^2 \dot{\Theta}_1^2 + \frac{1}{2} m R^2 \dot{\Theta}_2^2 - k R^2 (\Theta_2 - \Theta_1)^2 + \text{const}
\]

The equations of motion are:

\[
\begin{align*}
\Theta_1: & \quad m R^2 \ddot{\Theta}_1 - 2k R^2 (\Theta_2 - \Theta_1) = 0 \\
\Theta_2: & \quad m R^2 \ddot{\Theta}_2 + 2k R^2 (\Theta_2 - \Theta_1) = 0
\end{align*}
\]

or

\[
\begin{align*}
\ddot{\Theta}_1 + 2\omega_0^2 \Theta_1 - 2\omega_0^2 \Theta_2 &= 0 \\
\ddot{\Theta}_2 + 2\omega_0^2 \Theta_2 - 2\omega_0^2 \Theta_1 &= 0
\end{align*}
\]

In the matrix form this system of equations looks as follows:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\ddot{\Theta}_1 \\
\ddot{\Theta}_2
\end{pmatrix}
= -\begin{pmatrix}
2\omega_0^2 & -2\omega_0^2 \\
-2\omega_0^2 & 2\omega_0^2
\end{pmatrix}
\begin{pmatrix}
\Theta_1 \\
\Theta_2
\end{pmatrix}
\]

If we seek for the solution in the form

\[
\vec{\Theta} = \begin{pmatrix} \Theta_1(t) \\ \Theta_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i \omega t} = \vec{a} e^{i \omega t}
\]

then we obtain an eigenvalue problem

\[
K \vec{a} = \omega^2 M \vec{a}
\]

which has a nontrivial solution when \( \text{det}(K - \omega^2 M) = 0 \)

Thus,
\[ \begin{vmatrix} 2\omega_0^2 - \omega^2 & -2\omega_0^2 \\ -2\omega_0 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0 \quad \text{if } \lambda = \frac{\omega^2}{\omega_0^2} \text{ then we get} \\
(2-\lambda)^2 - 4 = 0 \quad \lambda_{1/2} = 4, 0 \]

Hence the roots are \( \omega^2_{1,2} = 4\omega_0^2, 0 \).

The first root, \( \omega^2 = 4\omega_0^2 \), yields the following eigenvector
\[
\begin{pmatrix}
-2\omega_0^2 \\
-2\omega_0^2
\end{pmatrix}
\begin{pmatrix}
a_1^{(1)} \\
a_2^{(1)}
\end{pmatrix}
= 0 \quad \Rightarrow \quad a_2^{(1)} = -a_1^{(1)} \quad \Rightarrow \quad \vec{a}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

So \( \Theta^{(1)}(t) = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2i\omega_0 t} \) where \( A \) is a constant.

The physically meaningful solution is, of course, obtained by taking the real part of \( \Theta^{(1)}(t) \).

The second root, \( \omega^2 = 0 \), needs to be treated with a little more care. Obviously
\[
\begin{pmatrix}
2\omega_0^2 \\
-2\omega_0^2
\end{pmatrix}
\begin{pmatrix}
a_1^{(2)} \\
a_2^{(2)}
\end{pmatrix}
= 0 \Rightarrow \quad a_2^{(2)} = a_1^{(2)} \quad \Rightarrow \quad \vec{a}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

The zero eigenvalue, \( \omega^2 = 0 \), does not correspond to the oscillatory motion. It corresponds to the equation
\[ \ddot{\xi}_2 = 0 \]
in normal coordinates \( \vec{\xi}' \), where \( \xi_2 = \frac{1}{12} \theta_1 + \frac{1}{12} \theta_2 \). The solution of that equation
\[ \xi_2(t) = \frac{1}{12} \theta_1(t) + \frac{1}{12} \theta_2(t) = \alpha t + \beta \quad \alpha, \beta = \text{const} \]
It corresponds to a motion where both beads rotate with a constant angular velocity.

The general solution of the equation of motion is
\[ \vec{\Theta}(t) = \vec{\Theta}^{(1)}(t) + \vec{\Theta}^{(2)}(t) = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2i\omega_0 t} + (Bt + C) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

where \( A \) is a complex constant and \( B \) and \( C \) are real constants. (effectively there are four real constants.)
a) Given the Lagrangian
\[ L = f(t) \left[ \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \right] \]
the equation of motion is
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \]
or
\[ \frac{d}{dt} \left( f(t) m \ddot{q} \right) - f(t) m \omega^2 q = 0 \]
or
\[ f \dddot{q} + f \ddot{q} - f \omega^2 q = 0 \]
or
\[ \dddot{q} + \frac{f}{f} \ddot{q} - \omega^2 q = 0 \]
Hence we see that \( \frac{f}{f} = 2\gamma \) and \( f(t) = e^{2\gamma t} \) \( (f(0) = 1) \)

b) \[ p = \frac{\partial L}{\partial \dot{q}} = f(t) m \dot{q} \quad \Rightarrow \quad \dot{q} = \frac{p}{mf} \]

Then the Hamiltonian is
\[ H = p \dot{q} - L(q, \dot{q}, p, \dot{p}, t) = \frac{p^2}{mf} - f \left[ \frac{1}{2} m \frac{p^2}{m^2} \dot{p}^2 - \frac{1}{2} m \omega^2 q^2 \right] = \]
\[ = \frac{p^2}{2m} e^{-2\gamma t} + \frac{m \omega^2 q^2}{2} e^{2\gamma t} \]

c) The original and transformed Hamiltonians are related via
\[ \hat{p} \hat{q} = H = p \dot{q} - H' + \frac{d}{dt} \left( F_2(q, p, t) - QP \right) \]
or, if we take the time derivative,
\[ \hat{p} \hat{q} = H = p \dot{q} - H' + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q} \dddot{q} + \frac{\partial F_2}{\partial p} \dddot{p} - \dot{Q} \dot{P} - Q \dot{P} \]

from which it follows that
\[ p = \frac{\partial F_2}{\partial q} \quad Q = \frac{\partial F_2}{\partial p} \quad H' = H + \frac{\partial F_2}{\partial t} \]

So when \( F_2 = e^{\gamma t} QP \) we obtain
\[ p = \frac{\partial F_2}{\partial q} = e^{\gamma t} P \quad Q = \frac{\partial F_2}{\partial p} = e^{\gamma t} q \quad \Rightarrow \quad q = e^{-\gamma t} Q \]

\[ H' = \frac{e^{2\gamma t} p^2}{2m} e^{2\gamma t} + \frac{m \omega^2 q^2}{2} e^{2\gamma t} + ye^{\gamma t} e^{-\gamma t} QP = \frac{p^2}{2m} + \frac{m \omega^2 q^2}{2} + yQ \]

\( H' \) does not have any explicit time dependence and is, therefore, conserved.