The Euler equations for a rigid body

Let us consider the torque-free motion of a rigid body. In such a case the potential energy $V$ vanishes and the Lagrangian $L$ becomes identical to the rotational kinetic energy. If we choose the coordinate system that correspond to the principal axes of rotation we have

$$T = \frac{1}{2} \sum_i I_i \omega_i^2$$  \((\star)\)

If we choose the Eulerian angles as the generalized coordinates, then the Lagrange's equation for $\psi$ is

$$\frac{\partial T}{\partial \psi} - \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\psi}} \right) = 0$$

It can also be expressed as

$$\sum_i \frac{\partial T}{\partial \dot{\omega}_i} \frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt} \left( \sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \psi} \right) = 0$$  \((\star\star)\)

If we differentiate $\dot{\omega}$ with respect to $\psi$ and $\dot{\psi}$ we get

$$\begin{align*}
\frac{\partial \omega_1}{\partial \psi} &= \dot{\theta} \sin \theta \cos \psi - \dot{\phi} \sin \phi = \omega_2 \\
\frac{\partial \omega_2}{\partial \psi} &= -\dot{\theta} \sin \theta \sin \psi - \dot{\phi} \cos \phi = -\omega_1 \\
\frac{\partial \omega_3}{\partial \psi} &= 0
\end{align*}$$

and

$$\frac{\partial \omega_1}{\partial \dot{\psi}} = \frac{\partial \omega_2}{\partial \dot{\psi}} = 0, \quad \frac{\partial \omega_3}{\partial \dot{\psi}} = 1$$

From $$(\star)$$ we also have \(\frac{\partial T}{\partial \omega_i} = I_i \omega_i$$

Then equation $$(\star\star)$$ becomes
\[ I_1 \omega_1 \omega_2 + I_2 \omega_2(\omega_1) - \frac{\text{d}}{\text{d}t} I_3 \omega_3 = 0 \]

or

\[ (I_1 - I_2) \omega_1 \omega_2 - I_3 \dot{\omega}_3 = 0 \]

By permuting indices 1,2,3 we can get relations for \( \dot{\omega}_1 \) and \( \dot{\omega}_2 \):

\[
\begin{align*}
(I_2 - I_3) \omega_2 \omega_3 - I_1 \dot{\omega}_1 &= 0 \\
(I_3 - I_1) \omega_3 \omega_1 - I_2 \dot{\omega}_2 &= 0 \\
(I_1 - I_2) \omega_1 \omega_2 - I_3 \dot{\omega}_3 &= 0
\end{align*}
\]

Euler's equation for a torque-free motion.

To obtain the Euler equations for the case when torques are present we start with

\[
\left( \frac{\text{d} \bar{L}}{\text{d}t} \right) = \bar{N} \quad \text{(prime stands for "fixed" frame)}
\]

In a previous lecture we also showed that

\[
\left( \frac{\text{d} \bar{L}}{\text{d}t} \right) = \left( \frac{\text{d} \bar{L}}{\text{d}t} \right) + \bar{\omega} \times \bar{L}
\]

no prime stands for "body" frame

Hence

\[
\frac{\text{d} \bar{L}}{\text{d}t} + \bar{\omega} \times \bar{L} = \bar{N}
\]

If we project this equation on the z-axis we get

\[
L_2 + \omega_x L_y - \omega_y L_x = N_z
\]

\[ (\text{x} \text{x} \text{x}) \]

However, since we have chosen the coordinate system in such a way that its axes coincide with the principal axes of the body we also have
\[ L_i = I_3 \omega_i \]

Then equation (**) becomes

\[ I \dot{\omega}_2 - (I_1 - I_2) \omega_1 \omega_2 = N_3 \]

We can either repeat this procedure manually and project the \[ \frac{d\mathbf{\omega}}{dt} + \mathbf{\omega} \times \mathbf{L} = \mathbf{N} \] equation onto \( y \) and \( x \) axes, or make thing more general by recalling that

\[ \mathbf{a} \times \mathbf{b} = \varepsilon_{ijk} a_j b_k \]

so that

\[ \mathbf{L} \times \mathbf{L} = N_{i} \quad i = 1, 2, 3 \]

With this the equations of motions are

\[ I_i \frac{d\omega_i}{dt} + \varepsilon_{ijk} \omega_j \omega_k I_k = N_i \quad i = 1, 2, 3 \]

Or, we can write them in the expanded form

\[ I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1 \]

\[ I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2 \]

\[ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3 \]