1. Find the Taylor series expansion of $\sin x$ at point $x = \pi/2$ up to the fourth order.

2. Find the value of the following integral when $a \to 0$:

$$\int_0^a \frac{x^2}{1 - \exp(-\beta x^2)} dx$$

(1) There are two ways to find the Taylor series. The first one is to follow the definition:

$$f(x-x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

In our case $f(x) = \sin x$, $x_0 = \frac{\pi}{2}$

Then

$$\sin^{(0)} \frac{\pi}{2} = 1$$
$$\sin^{(1)} \frac{\pi}{2} = \cos \frac{\pi}{2} = 0$$
$$\sin^{(2)} \frac{\pi}{2} = -\sin \frac{\pi}{2} = -1$$
$$\sin^{(3)} \frac{\pi}{2} = -\cos \frac{\pi}{2} = 0$$
$$\sin^{(4)} \frac{\pi}{2} = \sin \frac{\pi}{2} = 1$$

and

$$\sin x \approx \frac{x}{1!} - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - ...$$

The second and easier way is to recall that

$$\sin \left( x + \frac{\pi}{2} \right) = \cos x$$

When $x \to 0$ we know the expansion for $\cos x$:

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$$

Now we just replace $x \Rightarrow x-\frac{\pi}{2}$ and get

$$\sin x \approx \frac{x}{1!} - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - ...$$
(2) When \( a \to 0 \), \( x \) in the integrand is always small, so we can approximate it with its Maclaurin series and integrate each term separately:

\[
\frac{x^2}{1 - e^{-\beta x^2}} = \frac{x^2}{1 - (1 - \beta x^2 + \frac{\beta^2 x^4}{2!} - \frac{\beta^3 x^6}{3!} + \ldots)} = \frac{x^2}{\beta x^2 - \frac{\beta^2 x^4}{2} + \ldots}
\]

\[
\int_0^a \frac{x^2}{1 - e^{-\beta x^2}} \, dx \quad a \to 0 \quad \int_0^a \left( \frac{1}{\beta} + O(x^2) \right) \, dx = \frac{a}{\beta} + \mathcal{O}(a^3)
\]