PHYS 511: Computational Modeling and Simulation - Fall 2017
Assignment #3, due Monday November 13, by 5:00 pm

Solving the Poisson equation in 2D with the Galerkin method

In this assignment we will be solving the following Poisson equation in 2D using the Galerkin method:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) f(x,y) = -x+y^2, \tag{1}$$

The solution domain will be the square $0 \le x \le 1$, $0 \le y \le 1$. Function $f(x,y)$ is subject to the boundary conditions:

$$f(0,y)=0, \quad f(1,y)=0, \quad f(x,0)=0, \quad f(x,1)=0.$$

In the Galerkin aproach we need to choose suitable basis functions φ_i that satisfy the boundary conditions of the problem. In our case we can use the products of sine functions, i.e.

$$\varphi_{nm} = \sin(n\pi x)\sin(m\pi y), \qquad n=1,2,3,\dots \quad m=1,2,3,\dots,$$

as they conveniently take zero values at the edges of the simulation domain. With such a choice of the basis we will essentially be expanding the solution in a kind of a two-dimensional Fourier series

$$f_{\text{approx}} = \sum_{n=1}^{N}\sum_{m=1}^{M} c_{nm}\varphi_{nm} = \sum_{n=1}^{N}\sum_{m=1}^{M} c_{nm}\sin(n\pi x)\sin(m\pi y).$$

(to be precise, for any finite size of the basis this will not be a truncated Fourier series in the mathematical sense of that term as the algorithm for evaluating the expansion coefficients c_{nm} that we employ is completely different).

1. Write a Fortran program (`as3.f90`) that finds coefficients c_{nm} for a given size of the basis (you can have two integer constants that define N and M in your program). It should output these coefficients in a file called `c.dat` as a matrix, i.e.

$$\begin{matrix} c_{11} & c_{12} & c_{13} & \dots & c_{1M} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2M} \\ \vdots & & & & \\ c_{N1} & c_{N2} & c_{N3} & \dots & c_{NM} \end{matrix}$$

2. The program then should compute the numeric values of $f_{\text{approx}}(x,y)$ on a uniform square grid $0 \le x \le 1$, $0 \le y \le 1$ of $P \times Q$ points and output them into a data file that represents the solution (`solution.dat`), one grid point per line, i.e.

$$\begin{matrix} x_1 & y_1 & f(x_1,y_1) \\ x_1 & y_2 & f(x_1,y_2) \\ \vdots & & \\ x_N & y_M & f(x_N,y_M) \end{matrix}$$

3. Plot your numerical solution (e.g. with GNUPLOT or any other plotting software), generate a png file (`solution.png`) of sufficiently high resolution. Make sure everything is properly scaled, labelled, and readable.

4. Play around with changing N and M, make sure your calculations converge as you increase their values.

5. Submit your files (`as1.f90`, `c.dat`, `solution.dat`, `solution.png`, `report.txt`) for the case of $N = M = 20$ (i.e. 400 basis functions) and $P = Q = 200$ by uploading them to subdirectory `as3` shared with the instructor on your Google Drive.

Appendix: some useful integrals

In the expressions below k and l are assumed to be integers

$$\int_0^1 \sin(k\pi x)\sin(l\pi x)dx = \begin{cases} 1/2, & k = l \\ 0, & k \neq l \end{cases}$$

$$\int_0^1 \sin(k\pi x)dx = \frac{1+(-1)^{k+1}}{k\pi}$$

$$\int_0^1 x\sin(k\pi x)dx = \frac{(-1)^{k+1}}{k\pi}$$

$$\int_0^1 x^2\sin(k\pi x)dx = \frac{(-1)^k(2-k^2\pi^2)-2}{k^3\pi^3}$$